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The T_{+m} Transformation

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Abstract. This paper discusses a nonlinear sequence-to-sequence transformation, known as the T_{+m} transform, which is used to accelerate the convergence of an infinite series. A brief history of the transform is given; a number of theorems are established which enable one to make effective use of the transform, and several examples are presented to illustrate this effectiveness.

This paper discusses a nonlinear sequence-to-sequence transformation, known as the T_{+m} transformation, which is used to accelerate the convergence of an infinite series. A brief historical sketch of the transformation is given and a number of theorems are established which enable one to make effective use of the T_{+m} transform. In particular, special attention is given to the case in which the transformed series converges more rapidly or uniformly better than the original series. Finally, several examples are given to illustrate the effectiveness of the transformation.

The T_{+1} transformation was found independently by many authors. Generally, it was applied to some slowly convergent sequences for the purpose of accelerating their convergence. These sequences arose as a consequence of either (a) some iterative process or (b) as the partial sums of an infinite series.

Aitken [1] in 1926, Shanks and Walton [2] in 1948, Hartree [3] in 1949, and Isakson [4] in 1949 used the T_{+1} transformation in connection with iterative processes. On the other hand, Delaunay [5] in 1860, Samuelson [6] in 1945, Shanks [7], [8] in 1949 and 1955, and Lubkin [9] in 1952 applied the transformation to the sequence of partial sums of infinite series. With the exception of Aitken [1], Lubkin [9] and Shanks [7], [8], the transformation was simply applied in a particular problem(s) with little theoretical explanation being given of why it worked.

In 1969 Gray and Clark [10] defined and studied a generalization of the T_{+1} transformation which they called T_{+m}. They established some conditions under which the series generated by the T_{+m} transform would converge more rapidly than the original series.

Let $\{a(i)\}_{i=1}^{\infty}$ be an infinite sequence. Then the series associated with this sequence is denoted by $S = \Sigma_{i=1}^{\infty} a(i)$ and is the limit of the sequence $\{S(n)\}_{n=1}^{\infty}$ of partial sums $S(n) = \Sigma_{i=1}^{n} a(i)$.

Definition 1. The T_{+m} transformation is defined as

$$T_{+m}[S(n+m)] = \frac{S(n+m) - R(n;m)S(n)}{1 - R(n;m)}, \tag{1}$$

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where $m > 0$ is a positive integer and

$$R(n; m) = a(n + m)/a(n) \neq 1. \tag{2}$$

Another form of (1) which will be used extensively is

$$T_{+m}[S(n + m)] = S(n) + \frac{S(n + m) - S(n)}{1 - R(n; m)}. \tag{3}$$

Suppose that $\{A(n)\}_{n=1}^{\infty}$ and $\{B(n)\}_{n=1}^{\infty}$ are sequences of real numbers such that $\lim_{n\to\infty} A(n) = A$ and $\lim_{n\to\infty} B(n) = B$.

Definition 2. If

$$\lim_{n\to\infty}\left[\frac{A - A(n)}{B - B(n)}\right] = 0,$$

then $A(n)$ converges to A *more rapidly* than $B(n)$ converges to B.

Definition 3. If

$$\left|\frac{A - A(n)}{B - B(n)}\right| < 1 \quad \text{for all } n,$$

then $A(n)$ converges to A *uniformly better* than $B(n)$ converges to B.

These concepts are of great use in comparing rates of convergence of sequences. However, with respect to the T_{+m} transformation, their usefulness has been extremely limited because there exist few theorems which provide criteria that determine when the sequence $\{T_{+m}[S(n + m)]\}_{n=1}^{\infty}$ converges to S more rapidly or uniformly better than the sequence $\{S(n + m)\}_{n=1}^{\infty}$.

We begin our investigation with the following question: When does $\lim_{n\to\infty} T_{+m}[S(n + m)] = S$?

THEOREM 1. *If S is a convergent infinite series and* $\lim_{n\to\infty} |R(n; m)| = |R(m)| \neq 1$, *then* $T_{+m}[S(n + m)] \longrightarrow S$ *as* $n \longrightarrow \infty$.

Proof. From (3),

$$\lim_{n\to\infty} |T_{+m}[S(n + m)] - S(n)| = \lim_{n\to\infty}\left|\frac{S(n + m) - S(n)}{1 - R(n; m)}\right|$$

$$\leqslant \frac{1}{1 - |R(m)|} \lim_{n\to\infty} |S(n + m) - S(n)| = 0.$$

Thus $T_{+m}[S(n + m)] \longrightarrow S$ as $n \longrightarrow \infty$.

THEOREM 2. *Suppose that S is a convergent infinite series and there exists a constant* $C > 0$ *such that* $|1 - R(m; n)| > C$ *for n sufficiently large. Then* $T_{+m}[S(n + m)] \longrightarrow S$ *as* $n \longrightarrow \infty$.

Proof. It follows from (3) and the hypothesis that $|T_{+m}[S(n + m)] - S(n)| < C^{-1}\{|S(n + m) - S(n)|\} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus $T_{+m}[S(n + m)] \longrightarrow S$ as $n \longrightarrow \infty$.

Lubkin [9] proved the following theorem for the case $m = 1$. While this theorem is more general, its proof does not differ substantially from his and thus will be omitted.

THEOREM 3. *If the sequences* $\{S(n)\}_{n=1}^{\infty}$ *and* $\{T_{+m}[S(n + m)]\}_{n=1}^{\infty}$ *converge, then they have the same sum.*

THEOREM 4. *Suppose that S is an absolutely convergent infinite series whose terms, in absolute value, form a monotone decreasing sequence. If $\{n\,|a(n)|\}_{n=1}^{\infty}$ is a monotone nonincreasing sequence, then $T_{+m}[S(n+m)] \longrightarrow S$ as $n \longrightarrow \infty$.*

Proof. Since $\{n\,|a(n)|\}_{n=1}^{\infty}$ is a monotone nonincreasing sequence, then $(n+1)\,|a(n+1)| \leqslant n\,|a(n)|$ and

$$1/(1-|R(n;1)|) \leqslant n+1. \tag{4}$$

But $\{|a(n)|\}_{n=1}^{\infty}$ is a monotone decreasing sequence. Therefore,

$$\left|\frac{S(n+m)-S(n)}{1-R(n;m)}\right| \leqslant \frac{m\,|a(n)|}{1-|R(n;1)|};$$

and by (4), it follows that

$$\left|\frac{S(n+m)-S(n)}{1-R(n;m)}\right| \leqslant m(n+1)\,|a(n)|,$$

which converges to 0 as $n \longrightarrow \infty$.

Thus, applying (3), we have that $T_{+m}[S(n+m)] \longrightarrow S$ as $n \longrightarrow \infty$.

THEOREM 5. *Suppose that S is an absolutely convergent infinite series whose terms, in absolute value, form a monotone decreasing sequence. If $\{|R(n;m)|\}_{n=1}^{\infty}$ is a monotone nondecreasing sequence, then $T_{+m}[S(n+m)] \longrightarrow S$ as $n \longrightarrow \infty$.*

Proof. From (3) we have

$$|T_{+m}[S(n+m)] - S(n)| \leqslant \frac{|S(n+m)-S(n)|}{1-|R(n;m)|};$$

and thus,

$$|T_{+m}[S(n+m)] - S(n)| \leqslant \frac{m\,|a(n+1)|}{1-|R(n;1)|}. \tag{5}$$

Since $\{|a(n)|\}_{n=1}^{\infty}$ is a monotone decreasing sequence, then $|R(n;1)| < 1$, and so

$$\frac{|a(n+1)|}{1-|R(n;1)|} = \sum_{i=1}^{\infty} |a(n+1)|\,\{|R(n;1)|\}^{i-1}.$$

But this series is dominated, term-by-term, by the series $\Sigma_{i=n+1}^{\infty}|a(i)|$ which, since S is absolutely convergent, converges to 0 as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\to\infty} \frac{|a(n+1)|}{1-|R(n;1)|} = 0;$$

and thus, by (5), $T_{+m}[S(n+m)] \longrightarrow S$ as $n \longrightarrow \infty$.

We now consider the following question: When does the sequence generated by the T_{+m} transformation converge to S either more rapidly or uniformly better than the original sequence of partial sums?

THEOREM 6. *If S is a convergent infinite series with positive terms and $\lim_{n\to\infty}R(n;m) \neq 0$, then $T_{+m}[S(n+m)]$ converges to S more rapidly than $S(n+m)$.*

Proof. $T_{+m}[S(n+m)] \longrightarrow S$ by Theorem 1. By (3) and Definition 2, we have that

$$
\text{(6)}\quad
\begin{aligned}
\frac{T_{+m}[S(n+m)]-S}{S(n+m)-S} &= \frac{S(n)+\dfrac{S(n+m)-S(n)}{1-R(m;n)}-S}{S(n+m)-S}\\
&= \frac{S(n)-S}{S(n+m)-S}-\frac{1}{1-R(n;m)}\left[\frac{S(n+m)-S(n)}{S-S(n+m)}\right].
\end{aligned}
$$

By our hypothesis, $S(n)-S \longrightarrow 0$ and $S(n+m)-S \longrightarrow 0$ monotonely. Thus, by a theorem from Bromwich [11], we have

$$
\text{(7)}\quad
\begin{aligned}
\lim_{n\to\infty}\left[\frac{S(n)-S}{S(n+m)-S}\right] &= \lim_{n\to\infty}\left[\frac{S(n)-S-S(n-1)+S}{S(n+m)-S-S(n+m-1)+S}\right]\\
&= \lim_{n\to\infty}\left[\frac{a(n)}{a(n+m)}\right]=\lim_{n\to\infty}\left[\frac{1}{R(n;m)}\right].
\end{aligned}
$$

Similarly,

$$
\text{(8)}\qquad \lim_{n\to\infty}\left[\frac{S(n+m)-S(n)}{S-S(n+m)}\right]=\lim_{n\to\infty}\left[\frac{1}{R(n;m)}-1\right].
$$

Thus, substituting (7) and (8) into (6), it follows that

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\frac{T_{+m}[S(n+m)]-S}{S(n+m)-S}\right]\\
&\qquad=\lim_{n\to\infty}\left\{\frac{1}{R(n;m)}-\left[\frac{1}{1-R(n;m)}\right]\left[\frac{1}{R(n;m)}-1\right]\right\}=0,
\end{aligned}
$$

which completes the proof of the theorem.

THEOREM 7. *If S is a convergent series with positive monotone decreasing terms and $\{R(n;m)\}_{n=1}^{\infty}$ is a monotone nondecreasing sequence, then $T_{+m}[S(n+m)]$ converges to S uniformly better than $S(n+m)$ for all n.*

Proof. We first show that $\{T_{+m}[S(n+m)]\}_{n=1}^{\infty}$ is a monotone nondecreasing sequence in n; i.e., that $T_{+m}[S(n+m+1)] \geqslant T_{+m}[S(n+m)]$ for all n.

$$
\begin{aligned}
&T_{+m}[S(n+m+1)]-T_{+m}[S(n+m)]\\
&\quad=\frac{S(n+m+1)-R(n+1;m)S(n+1)}{1-R(n+1;m)}-\frac{S(n+m)-R(n;m)S(n)}{1-R(n;m)}\\
&\quad=a(n+1)+\frac{S(n+m+1)-S(n+1)}{1-R(n+1;m)}-\frac{S(n+m)-S(n)}{1-R(n;m)}\\
&\quad=\frac{a(n+1)[1-R(n+1;m)]+S(n+m+1)-S(n+1)}{1-R(n+1;m)}-\frac{S(n+m)-S(n)}{1-R(n;m)}\\
&\quad=[S(n+m)-S(n)]\left[\frac{1}{1-R(n+1;m)}-\frac{1}{1-R(n;m)}\right]\\
&\quad\geqslant 0 \quad \text{since } R(n+1;m)\geqslant R(n;m).
\end{aligned}
$$

Therefore, $T_{+m}[S(n+m+1)] \geqslant T_{+m}[S(n+m)]$ and so $\{T_{+m}[S(n+m)]\}_{n=1}^{\infty}$ is a monotone nondecreasing sequence in n.

We now note that $T_{+m}[S(n+m)] > S(n+m) > 0$ and by Theorem 5, $T_{+m}[S(n+m)] \longrightarrow S$ as $n \longrightarrow \infty$. But a positive nondecreasing sequence, if it converges, converges to its least upper bound. Therefore, for all n and any $m > 0$,

$$\left| \frac{S - T_{+m}[S(n+m)]}{S - S(n+m)} \right| = \frac{S - T_{+m}[S(n+m)]}{S - S(n+m)} < 1,$$

and so $T_{+m}[S(n+m)]$ converges to S uniformly better than $S(n+m)$.

To illustrate the above theory and the effectiveness of the T_{+m} transformation, we present the following examples. In the tables $E(n+m) = S - S(n+m)$ and $E_n(m) = S - T_{+m}[S(n+m)]$.

Example 1. Consider

$$S = \sum_{i=1}^{\infty} \frac{1}{i^3} \approx 1.20205690.$$

Choosing $m = 2$ we see, by Theorem 5, that $T_{+2}[S(n+2)]$ converges to S. Moreover, $T_{+2}[S(n+2)]$ converges to S uniformly better than $S(n+2)$ for all n by Theorem 7. The comparison of $T_{+2}[S(n+2)]$ with $S(n+2)$ is illustrated in Table 1.

Example 2. Consider

$$S = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i-1} = \frac{\pi}{4} \approx .78539816.$$

In this example, we choose $m = 1$.

It takes approximately 50,000 terms of this series to obtain four digit accuracy. However, from Table 2, we see that $T_{+1}[S(n+1)]$ achieves this same degree of accuracy from the first nine terms of the series.

Example 3. Consider

$$S = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = \ln 2 \approx .69314718.$$

It can be shown that it takes 10,000 terms of this series to obtain seven digit accuracy. However, choosing $m = 1$ and examining Table 3, we find that this same degree of accuracy can be obtained from the first fourteen terms of the series by means of $T_{+1}[S(n+1)]$.

n	$S(n+2)$	$E(n+2)$	$T_{+2}[S(n+2)]$	$E_n(2)$
10	1.1988618	.0031951	1.2010725	.0009844
20	1.2010690	.0009879	1.2017394	.0003175
30	1.2015825	.0004744	1.2019013	.0001556
40	1.2017786	.0002783	1.2019642	.0000927
50	1.2019735	.0001834	1.2019966	.0000603

TABLE 1. Application of $T_{+2}[S(n+2)]$ to $S = \Sigma_{i=1}^{\infty} \frac{1}{i^3}$

n	$S(n+1)$	$E(n+1)$	$T_{+1}[S(n+1)]$	$E_n(1)$
1	.8666666	−.0812684	.7916666	−.0062684
2	.7238095	.0615887	.7833333	.0020649
3	.8349206	−.0495224	.7863095	−.0009113
4	.7440115	.0413867	.7849206	.0004776
5	.8209346	−.0355364	.7856782	−.0002800
6	.7542679	.0311303	.7852203	.0001779
7	.8130915	−.0276933	.7855179	−.0001197
8	.7604599	.0249383	.7853139	.0000843

TABLE 2. Application of $T_{+1}[S(n+1)]$ to $S = \Sigma_{i=1}^{\infty} \dfrac{(-1)^{i+1}}{2i-1}$

n	$S(n+1)$	$E(n+1)$	$T_{+1}[S(n+1)]$	$E_n(1)$
3	.77777776	−.08463058	.65333333	.03981385
5	.77089945	−.07775227	.68342151	.00972567
7	.75506524	−.06191806	.68951776	.00362942
9	.74454215	−.05139497	.69233120	.00081598
11	.73627423	−.04312705	.69300004	.00014714
13	.73006693	−.03691975	.69314716	.00000002

TABLE 3. Application of $T_{+1}[S(n+1)]$ to $S = \Sigma_{i=1}^{\infty} \dfrac{(-1)^{i+1}}{i}$

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